

MORE ABSORBERS IN HYPERSPACES

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ABSTRACT. The family of all subcontinua that separate a compact connected n -manifold X (with or without boundary), $n \geq 3$, is an F_σ -absorber in the hyperspace $C(X)$ of nonempty subcontinua of X . If $D_2(F_\sigma)$ is the small Borel class of spaces which are differences of two σ -compact sets, then the family of all $(n-1)$ -dimensional continua that separate X is a $D_2(F_\sigma)$ -absorber in $C(X)$. The families of nondegenerate colocally connected or aposyndetic continua in I^n and of at least two-dimensional or decomposable Kelley continua are $F_{\sigma\delta}$ -absorbers in the hyperspace $C(I^n)$ for $n \geq 3$. The hyperspaces of all weakly infinite-dimensional continua and of C -continua of dimensions at least 2 in a compact connected Hilbert cube manifold X are Π_1^1 -absorbers in $C(X)$. The family of all hereditarily infinite-dimensional compacta in the Hilbert cube I^ω is Π_1^1 -complete in 2^{I^ω} .

1. INTRODUCTION

The theory of absorbing sets was well developed in the eighties and nineties of the last century (see [1] and [24]). Since any two absorbers in a Hilbert cube \mathcal{Q} of a given Borel or projective class are homeomorphic via arbitrarily small ambient homeomorphisms of \mathcal{Q} , it provides a powerful technique of characterizing some subspaces of the hyperspaces 2^X of all closed nonempty subsets of a nondegenerate Peano continuum X or $C(X)$ of all subcontinua of X (if X contains no free arcs). Nevertheless, the list of natural examples of such spaces is not too long and they usually are considered in X being a Euclidean or the Hilbert cube.

Let $I = [0, 1]$ be the closed unit interval with the Euclidean metric. The following subspaces of a respective hyperspace are absorbers of Borel class F_σ (otherwise known as cap-sets), so they are homeomorphic to the pseudo-boundary $B(I^\omega) = \{(x_i) \in I^\omega : \exists i (x_i \in \{0, 1\})\}$, a standard F_σ -absorber in the Hilbert cube I^ω :

Examples 1.1.

- (1) The subspace $\mathcal{D}_n(X)$ of 2^X consisting of all compacta of covering dimensions $\geq n \geq 1$, where X is a locally connected continuum each of whose open non-empty subset has dimension $\geq n$ [5] (for $X = I^\omega$, see [11]),
- (2) The family of all decomposable continua in I^n , $n \geq 3$ [28],
- (3) The family of all compact subsets (subcontinua) with nonempty interiors in a locally connected nondegenerate continuum (containing no free arcs) [9],
- (4) The family of all compact subsets that block all subcontinua of a locally connected nondegenerate continuum which is not separated by any finite subset [15].

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Examples 1.2. Known $F_{\sigma\delta}$ -absorbers include two standard ones $(B(I^\omega))^\omega$ (in $(I^\omega)^\omega$) and $\widehat{c}_0 = \{(x_i) \in I^\omega : \lim_i x_i = 0\}$ (in I^ω) [11] and

- (1) The subspace of 2^{I^ω} of all infinite-dimensional compacta [11],
- (2) The subspace of $C(I^n)$ of all locally connected subcontinua of I^n , $n \geq 3$ [14],
- (3) The subspaces of $C(I^2)$ of all arcs [4] and of all absolute retracts [6].

Examples 1.3. If $D_2(F_\sigma)$ is the class of all subsets of the Hilbert cube that are differences of two F_σ -sets, then a standard $D_2(F_\sigma)$ -absorber is the subset $B(I^\omega) \times s$ of $I^\omega \times I^\omega$ ($s = I^\omega \setminus B(I^\omega)$); its incarnation in 2^{I^ω} is the family $\mathcal{D}_n(I^\omega) \setminus \mathcal{D}_{n+1}(I^\omega)$, $n \geq 1$ [11].

Example 1.4. The subspace of $2^{\mathbb{R}^n}$, $n \geq 3$, of all compact ANR's in \mathbb{R}^n is a $G_{\delta\sigma\delta}$ -absorber [12].

Let Π_1^1 and Σ_1^1 denote the classes of coanalytic and analytic sets, respectively. Concerning Π_1^1 -absorbers (coanalytic absorbers), the Hurewicz set \mathcal{H} of all countable closed subset of I can be treated as a standard Π_1^1 -absorber in the hyperspace 2^I [3]. Other known examples of coanalytic absorbers we wish to recall here are:

Examples 1.5.

- (1) The subspace of $C(I^n)$, $3 \leq n \leq \omega$, consisting of all hereditarily decomposable continua [27],
- (2) The subspaces \mathcal{SCD}_k of 2^{I^ω} of all strongly countable-dimensional compacta of dimensions $\geq k$ and $\mathcal{SCD}_{k+1} \cap C(I^\omega)$ of $C(I^\omega)$ of continua of dimensions $> k$, for any $k \in \mathbb{N}$ [20],
- (3) The families of Wilder continua, of continuum-wise Wilder continua and of hereditarily arcwise connected nondegenerate continua in I^n , $n \geq 3$ [17].

The main purpose of this paper is to add to the list some new examples of absorbers in the hyperspaces of cubes, n -manifolds or the Hilbert cube manifolds.

In Section 4 we deal with three important in continuum theory classes Col of colocally connected, Apo of aposyndetic and \mathcal{K} of Kelley continua. Class Apo properly contains Col and it is properly contained in the class of, so called, continua with property C [29] which we call Wilder continua [17]. The class LC of locally connected continua (= Peano continua) is properly contained in both Apo and \mathcal{K} . We evaluate the Borel class of families $Apo(Y)$ and $Col(Y)$ of nondegenerate aposyndetic and colocally connected, resp., subcontinua of a compact space Y as $F_{\sigma\delta}$ in hyperspace $C(Y)$ and show that the families are $F_{\sigma\delta}$ -universal if Y contains a 2-cell. An analogous result is known for the family $\mathcal{K}(Y)$ of Kelley subcontinua of a Peano continuum Y [19, Theorem 3.3]. We prove that if Y is a Peano continuum each of whose open subset contains an n -cell ($n \geq 2$), then $Apo(Y)$, $\mathcal{K}(Y)$ and $LC(Y)$ are strongly $F_{\sigma\delta}$ -universal in $C(Y)$ and the families restricted to at least 2-dimensional continua are $F_{\sigma\delta}$ -absorbers in $C(Y)$. Moreover, $Col(I^n)$ and $Apo(I^n)$ are shown to be $F_{\sigma\delta}$ -absorbers in $C(I^n)$, $n \geq 3$. The same proof shows that also the family of all decomposable Kelley continua in I^n are $F_{\sigma\delta}$ -absorbers in $C(I^n)$ for $n \geq 3$.

In Section 5 we show that if X is a locally connected continuum such that each open non-empty subset of X contains a copy of $(0, 1)^n$, $3 \leq n < \infty$, as an open subset and no subset of dimension ≤ 1 separates X , then the families $\mathcal{S}(X)$ of all closed separators of X and $\mathcal{S}(X) \cap C(X)$ of all continua that separate X are F_σ -absorbers in 2^X and $C(X)$, respectively. Nowhere dense closed separators and

nowhere dense continua which separate X from $D_2(F_\sigma)$ -absorbers in 2^X and $C(X)$, respectively.

One of the central notions in the theory of infinite dimension is that of a weakly infinite-dimensional space introduced by P. S. Alexandroff in 1948. In Section 6 we find two new coanalytic absorbers in 2^X ($C(X)$) for each locally connected continuum X such that each open non-empty subset of X contains a Hilbert cube: the family of weakly infinite-dimensional compacta (continua) in X of dimensions $\geq n \geq 1$ ($\geq n \geq 2$) and the family of compacta (continua) which are C -spaces of dimensions $\geq n \geq 1$ ($\geq n \geq 2$).

We observe that the collection of hereditarily infinite-dimensional compacta in I^ω is a Π_1^1 -complete subset of 2^{I^ω} contained in a σZ -set. We do not know however if it is a coanalytic absorber in 2^{I^ω} .

2. PRELIMINARIES

All spaces in the paper are assumed to be metric separable. The hyperspace 2^X of nonempty compact subsets of X is endowed with the Hausdorff metric *dist* and the hyperspace $C(X)$ of continua in X is considered as a subspace of 2^X .

If X is a locally connected nondegenerate continuum, then 2^X is homeomorphic to I^ω and if, additionally, X contains no free arcs, then also $C(X)$ is a Hilbert cube [10].

A closed subset C *separates* a space X between two disjoint subsets A and B if there are open disjoint subsets U and V of X such that $A \subset U$, $B \subset V$ and $X \setminus C = U \cup V$. Such set C will be called a *closed separator* in X .

A subset C *cuts* X between two disjoint subsets A and B (C is a *cut* between A and B) if it is disjoint from $A \cup B$ and any continuum $D \subset X$ that meets both A and B intersects C .

If a compact space X is locally connected, then a closed subset $C \subset X$ separates X between A and B if and only if C cuts X between A and B [22, Theorem 1, p. 238].

Suppose X is a closed subset of a space Y . It is known that for each closed separator C in X between disjoint closed subsets A and B of X , there is a closed separator C' in Y between A and B such that $C' \cap X = C$ [13, Lemma 1.2.9, Remark 1.2.10].

Combining these two classic facts, we get an easy observation.

Observation 2.1. *Let X be a closed subset of a compact locally connected space Y . A closed set $C \subset X$ separates X between closed subsets A and B of X if and only if C' cuts Y between A and B .*

Recall that, given a class \mathcal{M} of spaces, a subset A of a complete space Z is \mathcal{M} -complete if $A \in \mathcal{M}$ and A is \mathcal{M} -hard, i.e., for any subset $C \in \mathcal{M}$ of a complete 0-dimensional space Y there is a continuous mapping $\xi : Y \rightarrow Z$ (called a *reduction of C to A*) such that $\xi^{-1}(A) = C$ [16].

A closed subset B of a Hilbert cube (\mathcal{Q}, d) is a Z -set in \mathcal{Q} if for any $\epsilon > 0$ there exists a continuous mapping $f : \mathcal{Q} \rightarrow \mathcal{Q}$ such that $f(\mathcal{Q}) \cap B = \emptyset$ and $\tilde{d}(f, \text{id}_{\mathcal{Q}}) = \sup\{d(f(x), x) : x \in \mathcal{Q}\} < \epsilon$. A countable union of Z -sets in \mathcal{Q} is called a σZ -set in \mathcal{Q} .

Let \mathcal{M} be a class of spaces which is topological (i.e., if $M \in \mathcal{M}$ then each homeomorphic image of M belongs to \mathcal{M}) and closed hereditary (i.e., each closed

subset of $M \in \mathcal{M}$ is in \mathcal{M}). Following [11], we call a subset A of a Hilbert cube \mathcal{Q} \mathcal{M} -universal if for each $M \subset I^\omega$ from the class \mathcal{M} there is an embedding $f : I^\omega \rightarrow \mathcal{Q}$ (a reduction of M to A) such that $f^{-1}(A) = M$; A is said to be *strongly \mathcal{M} -universal* if for each $M \subset I^\omega$ from the class \mathcal{M} and each compact set $K \subset I^\omega$, any embedding $f : I^\omega \rightarrow \mathcal{Q}$ such that $f(K)$ is a Z -set in \mathcal{Q} can be approximated arbitrarily closely by an embedding $g : I^\omega \rightarrow \mathcal{Q}$ such that $g(I^\omega)$ is a Z -set in \mathcal{Q} , $g|_K = f|_K$ and $g^{-1}(A) \setminus K = M \setminus K$.

Observe that a strongly \mathcal{M} -universal set is \mathcal{M} -universal and an \mathcal{M} -universal set is \mathcal{M} -hard. Often in practice, in order to show that a set $A \subset \mathcal{Q}$ is \mathcal{M} -universal (or \mathcal{M} -hard), we choose an already known \mathcal{M} -universal (\mathcal{M} -hard) set $B \subset I^\omega$ and construct a continuous embedding $\xi : I^\omega \rightarrow \mathcal{Q}$ such that $\xi^{-1}(A) = B$.

A subset A of a Hilbert cube \mathcal{Q} is called an \mathcal{M} -absorber in \mathcal{Q} provided that:

- (1) $A \in \mathcal{M}$;
- (2) A is contained in a σZ -set in \mathcal{Q} ;
- (3) A is strongly \mathcal{M} -universal.

3. PROVING STRONG \mathcal{M} -UNIVERSALITY IN HYPERSPACES

We are now going to sketch two techniques for proving strong \mathcal{M} -universality. The first one, developed in [14], [27, Lemma 3.2], applies to subsets \mathcal{A} of 2^{I^n} or $C(I^n)$, $n \in \mathbb{N} \cup \{\omega\}$ (for simplicity we will consider the case $n \in \mathbb{N}$). The second, presented in [5], concerns subsets \mathcal{A} of the Hilbert cube $\mathcal{Q} = 2^X$ or $\mathcal{Q} = C(X)$ in a much more general case of a locally connected nondegenerate continuum X (without free arcs) satisfying certain local properties. For some classes of continua located in such X the second technique may fail while the first one still works if X is a cube.

3.1. Approach I. Suppose that K is a compact subset of I^ω and $f : I^\omega \rightarrow C(I^n)(2^{I^n})$ is an embedding such that $f(K)$ is a Z -set and $\epsilon > 0$. We have to find a Z -embedding $g : I^\omega \rightarrow C(I^n)(2^{I^n})$ which agrees with f on K , is ϵ -close to f and satisfies

$$(3.1) \quad g^{-1}(\mathcal{A}) \setminus K = M \setminus K.$$

For a construction of g we need an auxiliary map $\theta : I^\omega \rightarrow C([-1, 1]^n)$ sending $q = (q_i)$ to

$$\theta(q) = \left(\left([-1, 0] \times \{0\} \right) \cup S\left(-\frac{1}{2}, 0; \frac{1}{2}\right) \cup \bigcup_{i=1}^{\infty} S(a_i; r_i(q)) \right) \times \{(0, \dots, 0)\},$$

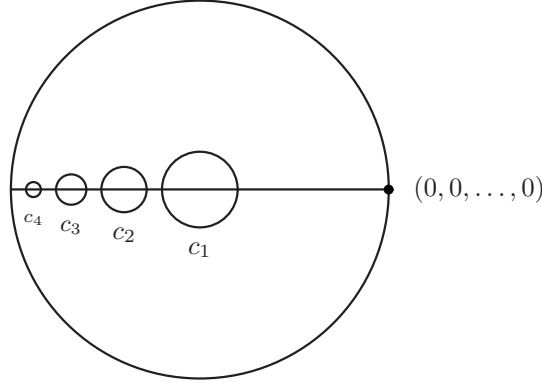
where $S(x; r)$ denotes the circle in the plane centered at x with radius r , $a_i = (-1 + 2^{-i}, 0) \in \mathbb{R}^2$ and $r_i(q) = 4^{-(i+1)}(1 + q_i)$ (see Figure 1).

The set $\theta(q)$ is the union of disjoint circles $c_i = S(a_i; r_i(q))$ contained in $[-1, 0] \times [-1, 1] \times \{(0, \dots, 0)\}$ and of the diameter segment of the largest circle. The inner

circles c_i uniquely code the point (q_i) and the map θ is a continuous embedding.

We also will exploit a continuous deformation $H_0 : 2^{I^n} \times I \rightarrow 2^{I^n}$ through finite sets such that, for any $(A, t) \in 2^{I^n} \times (0, \frac{1}{2}]$, $H_0(A, t)$ is finite,

$$\text{dist}(A, H_0(A, t)) \leq 2t \quad \text{and} \quad H_0(A, t) \subset [t, 1 - t]^n.$$


 FIGURE 1. $\theta(q)$

Connecting points of $H_0(A, t)$, $t > 0$, one can define another deformation through finite graphs

$$(3.2) \quad H(A, t) = \begin{cases} \bigcup_{a, b \in H_0(A, t)} (\overline{ab} \cap (\overline{B}(a; 2t) \cup \overline{B}(b; 2t))) & \text{if } t > 0, \\ A & \text{if } t = 0, \end{cases}$$

where $\overline{B}(a; \alpha)$ is the closed α -ball in I^n around a and \overline{ab} is the line segment in I^n from a to b . For any $(A, t) \in C(I^n) \times (0, \frac{1}{2})$, $H(A, t)$ is a connected graph in $[t, 1 - t]^n$ and $\text{dist}(A, H(A, t)) \leq 4t$ (see [14] and [27] for details).

Assume now that, for each subset $M \subset I^\omega$ which belongs to class \mathcal{M} , there exists a continuous map $\xi : I^\omega \rightarrow C(I^n)(2^{I^n})$ such that $\xi^{-1}(\mathcal{A}) = M$ and $(0, 0, \dots) \in \xi(x)$ for every $x \in I^\omega$.

An embedding g can be defined in the form

$$(3.3) \quad g(q) = H(f(q), \mu(q)) \cup \bigcup_{x \in H_0(f(q), \mu(q))} (x + \mu(q)\theta(q)) \cup \bigcup_{x \in H_0(f(q), \mu(q))} (x + \mu(q)\xi(q))$$

(we use linear operations of addition and scalar multiplication in (3.3)), where

$$\mu(q) = \frac{1}{12} \min\{\epsilon, \min\{\text{dist}(f(q), f(z)) : z \in K\}\}.$$

If one is interested in the strong \mathcal{M} -universality of \mathcal{A} in 2^{I^n} then the “graph” part $H(f(q), \mu(q))$ above is skipped. In Section 4, in the proof of Theorem 4.3 we modify g by replacing $H(f(q), \mu(q))$ with the closed ball $\overline{B}(H(f(q), \mu(q)); \frac{1}{8}\mu(q))$.

One can check that g is always a Z -embedding which agrees with f on K and ϵ -approximates f (see the proof of [27, Lemma 3.2]; some details of it are sketched in the proof of Theorem 4.3). So, it only remains to verify property (3.1).

3.2. Approach II. Assume X is a nondegenerate locally connected continuum without free arcs. As before, we are looking for a Z -embedding g that approximates f , agrees with f on K and satisfies (3.1), where the cube I^n is replaced with X . Suppose that for each non-empty open subset U of X there is a continuous mapping $\varphi_U : I^\omega \rightarrow C(U)$ such that $\varphi_U^{-1}(\mathcal{A}) = M$. If X is a Peano continuum then each

φ_U can be extended to a continuous mapping $\tilde{\varphi}_U : I^\omega \times I \rightarrow C(U)$ such that $\tilde{\varphi}_U(q, 1) = \varphi_U(q)$ and $\tilde{\varphi}_U(q, t)$ is a finite union of arcs in U for all $t \in (0, 1]$ and $q \in I^\omega$. If U contains a homeomorphic image of $(0, 1)^n$, $n \geq 2$, then we can put $\tilde{\varphi}_U(q, t) = \zeta_U(H(\varphi_U(q), 1 - t))$ where $\zeta_U : (0, 1)^n \rightarrow U$ is an embedding and H is the homotopy defined in (3.2). In this case $\tilde{\varphi}_U(q, t)$ is a finite graph whenever $t \in (0, 1]$.

Without repeating complicated details of the construction in [5, Section 5], it is enough for our purposes to remark that the image $g(q)$, for $q \in I^\omega \setminus K$, is of the form $A(q) \cup \bigcup_{U \in S(q)} \tilde{\varphi}_U(q)$, where $A(q)$ is a compactum in X , $S(q)$ is a finite set, $\tilde{\varphi}_U(q) = \tilde{\varphi}_U(q, t(q))$ and $\tilde{\varphi}_U(q) = \varphi_U(q)$ for at least one $U \in S(q)$. We can also assume that distinct $U, U' \in S(q)$ are disjoint. In the case $\mathcal{A} \subset 2^X$, $A(q)$ is at most 1-dimensional. If $\mathcal{A} \subset C(X)$ then $g(q)$ is a continuum and $A(q)$ is a 1-dimensional locally connected continuum which is a countable union of arcs and each $\tilde{\varphi}_U(q)$, $U \in S(q)$, intersects $A(q)$. Moreover, if each non-empty open subset of X has dimension ≥ 2 , then we can also assume that the intersection $\tilde{\varphi}_U(q) \cap A(q)$ is a singleton for each $U \in S(q)$. The construction guarantees that g is a Z -embedding.

We will use this approach and verify (3.1) in Sections 5 and 6, and partly in 4.

4. COLOCALLY CONNECTED, APOSYNDETTIC AND KELLEY CONTINUA

Recall that a continuum X is *aposyndetic* if and only if X is semi-locally connected, i.e. for any $\epsilon > 0$, each point $x \in X$ has an open neighborhood U of diameter $\text{diam } U < \epsilon$ such that $X \setminus U$ has finitely many components; if one requires that $X \setminus U$ be connected then X is *colocally connected*.

Proposition 4.1. *If Y is a compact space, then the families $\text{Apo}(Y)$ and $\text{Col}(Y)$ of nondegenerate aposyndetic continua and colocally connected continua in Y , resp., are $F_{\sigma\delta}$ -subsets of $C(Y)$. If Y contains a copy of I^2 , then the families are $F_{\sigma\delta}$ -universal.*

Proof. We first evaluate the Borel class of $\text{Col}(Y)$. By compactness, $X \in \text{Col}(Y)$ if and only if for each $\epsilon > 0$ there is a finite ϵ -cover of X consisting of open subsets of X with connected complements. Passing to complements, this can be written in terms of closed subsets of Y as follows:

$$(4.1) \quad X \in \text{Col}(Y) \quad \text{if and only if} \\ |X| > 1 \quad \wedge \quad (\forall n) (\exists m) (\exists K_1, \dots, K_m \in C(Y)) \\ \bigcap_{i=1}^m K_i = \emptyset \quad \wedge \quad (\forall i \leq m) \left(K_i \subset X \quad \wedge \quad \text{diam}(X \setminus K_i) < \frac{1}{n} \right).$$

Since formula (4.1) yields merely analyticity of $\text{Col}(Y)$, it needs further refinements. It seems more convenient to deal with the complement of $\text{Col}(Y)$ in $C(Y)$. Recall that the function

$$f : C(Y) \times C(Y) \rightarrow 2^Y, \quad f(X, K) = \overline{X \setminus K}$$

is lower semi-continuous [21, p. 182]. Hence, the function

$$(X, K) \mapsto \text{diam}(f(X, K))$$

is of the first Borel class [22, Theorem 1, p. 70] which yields that the set

$$\begin{aligned} \{ (X, K) \in C(Y) \times C(Y) : \text{diam}(X \setminus K) \geq \frac{1}{n} \} = \\ \{ (X, K) \in C(Y) \times C(Y) : \text{diam}(\overline{X \setminus K}) \geq \frac{1}{n} \} \end{aligned}$$

is G_δ in $C(Y) \times C(Y)$ for each n . Since $X \in C(Y) \setminus Col(Y)$ if and only if

$$(4.2) \quad |X| = 1 \quad \vee \quad (\exists n) (\forall m) (\forall K_1, \dots, K_m \in C(Y)) \\ \bigcap_{i=1}^m K_i \neq \emptyset \quad \vee \quad (\exists i \leq m) \left(K_i \not\subset X \quad \vee \quad \text{diam}(X \setminus K_i) \geq \frac{1}{n} \right),$$

we get that $C(Y) \setminus Col(Y)$ is $G_{\delta\sigma}$ in $C(Y)$.

The proof for the family $Apo(Y)$ is similar.

Passing to the second part, we can assume that Y contains I^2 . Let (J_{ij}) , $i, j = 1, 2, \dots$, be a double sequence of mutually disjoint nondegenerate closed intervals in I such that the length of J_{ij} is less than $\frac{1}{4^{i+j}}$ and, for each j , $\text{Lim}_i J_{ij} \rightarrow \{1\}$. Consider rectangles

$$R_{ij}(t) = J_{ij} \times \left[0, \frac{t}{j+1} \right],$$

and define a continuous embedding $\psi : I^\omega \rightarrow C(Y)$ by

$$(4.3) \quad \psi((q_i)) = \partial(I^2) \cup \bigcup_{i,j} R_{ij}(q_i)$$

(Figure 2; instead of rectangles one can also use their boundaries).

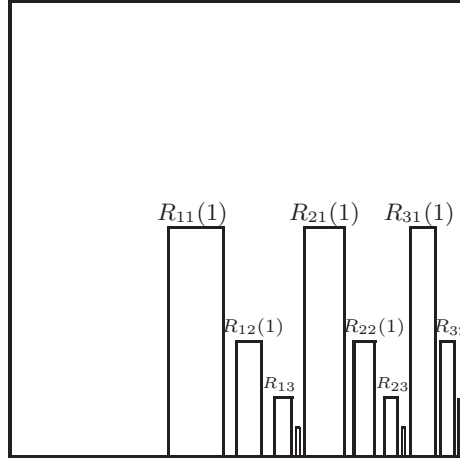


FIGURE 2. $\psi(1, 1, 1, \dots)$

It satisfies

$$(4.4) \quad \psi^{-1}(Col(Y)) = \psi^{-1}(Apo(Y)) = \hat{c}_0$$

and since \hat{c}_0 is an $F_{\sigma\delta}$ -absorber in I^ω , the proof is complete. \square

Proposition 4.2. *If $D(Y)$ denotes the family of all decomposable continua in a space Y , then $Col(Y) \subset Apo(Y) \subset D(Y)$ and $D(I^n)$ is a σZ -set in $C(I^n)$ for $n \geq 3$.*

Proof. It is known that each nondegenerate aposyndetic continuum is decomposable. The last part of the proposition was proved in [27]. \square

Theorem 4.3. *$Apo(I^n)$ and $Col(I^n)$ are $F_{\sigma\delta}$ -absorbers in $C(I^n)$ for $n \geq 3$.*

Proof. Concerning $Apo(I^n)$, we can refer to the proof of the strong $F_{\sigma\delta}$ -universality of the family of all Peano continua $LC(I^n)$ due to Gladdines and van Mill [14]. It perfectly works for $Apo(I^n)$. Below, we sketch an alternative construction that suits both families. Actually, we appropriately modify the construction of the embedding g from Subsection 3.1.

First, locate continua $\psi((q_i))$ from (4.3) in I^n as

$$\psi'((q_i)) = \psi((q_i)) \times \{(0, \dots, 0)\}.$$

$n-2$

The set \hat{c}_0 being an $F_{\sigma\delta}$ -absorber in I^ω , there is, for an $F_{\sigma\delta}$ -set $M \subset I^\omega$, a mapping $\zeta : I^\omega \rightarrow I^\omega$ such that $\zeta^{-1}(\hat{c}_0) = M$. Put

$$(4.5) \quad \xi = \psi' \zeta.$$

Second, since the “graph” ingredient $H(f(q), \mu(q))$ may spoil the colocal connectedness of $g(q)$, we surround it by a closed ball

$$\overline{B}(H(f(q), \mu(q)); \frac{1}{8}\mu(q))$$

of a small enough radius (e. g., $\frac{1}{8}\mu(q)$) and redefine the embedding as follows:

$$(4.6) \quad g'(q) = \overline{B}(H(f(q), \mu(q)); \frac{1}{8}\mu(q)) \cup \bigcup_{x \in H_0(f(q), \mu(q))} (x + \mu(q)\theta(q)) \cup \bigcup_{x \in H_0(f(q), \mu(q))} (x + \mu(q)\xi(q)).$$

Clearly, g' is an ϵ -approximation of f and $g'|K = f|K$. So, it is 1-to-1 on K . For distinct $q, q' \in I^\omega \setminus K$, coefficients $\mu(q)$ and $\mu(q')$ are positive. Let x, x' be the minimal points in $H_0(f(q), \mu(q))$ and $H_0(f(q'), \mu(q'))$, respectively, in the lexicographic order on I^n . Then $g'(q) \neq g'(q')$ because the inner circles in copies $x + \mu(q)\theta(q)$ and $x' + \mu(q')\theta(q')$ remain disjoint from the rest of $g'(q)$ and $g'(q')$, respectively. If $q \in K, q' \notin K$, then $g'(q) \neq g'(q')$ by a similar simple estimation of $dist(g'(q), g'(q'))$ as for the original approximation g . Moreover, $g'(K) = f(K)$ is a Z -set in $C(I^n)$ by assumption and $g'(I^\omega) \setminus g'(K)$, as an open subset of $g'(I^\omega)$, is F_σ in $C(I^n)$. Notice that $g'(q)$, for each $q \in I^\omega \setminus K$, contains an open in $g'(q)$, one-dimensional subset of $x + \mu(q)\theta(q)$ (e. g., the inner circles in there). Therefore, the deformation

$$h : C(I^n) \times I \rightarrow C(I^n), \quad h(A, t) = \overline{B}(A; t)$$

maps $g'(I^\omega \setminus K)$ off the set $I^\omega \setminus K$ arbitrarily closely to the identity map on $C(I^n)$. It means that $g'(I^\omega) \setminus g'(K)$ is a σZ -set and, consequently, $g'(I^\omega)$ is a Z -set in $C(I^n)$.

Finally, we need to check the property

$$(4.7) \quad (g')^{-1}(Col(I^n)) \setminus K = (g')^{-1}(Apo(I^n)) \setminus K = M \setminus K.$$

So, suppose $q \notin K$. Then $\mu(q) > 0$. If $q \in M$, then one can easily see that $g'(q) \in Col(I^n)$. If $q \notin M$, it is convenient to consider the maximal point $y = (y_1, y_2, \dots, y_n) \in H_0(f(q), \mu(q))$ in the lexicographic order \prec on I^n . The copy $y + \mu(q)\xi(q)$ of $\psi(\zeta(q))$ is not aposyndetic by (4.4) and it can be intersected by at most finitely many other isometric copies $x + \mu(q)\xi(q)$, $x \in H_0(f(q), \mu(q))$, where $x = (x_1, x_2, y_3, \dots, y_n) \prec y$. Neither these copies nor adding the part

$$\overline{B}(H(f(q), \mu(q)); \frac{1}{8}\mu(q)) \cup \bigcup_{x \in H_0(f(q), \mu(q))} (x + \mu(q)\theta(q))$$

affect the non-semi-local connectedness caused by $y + \mu(q)\xi(q)$. So, $g'(q) \notin Apo(I^n)$ and (4.7) is satisfied. \square

Recall that a continuum Y is called a *Kelley continuum* if for each point $z \in Y$, each sequence of points $z_n \in Y$ converging to z and each subcontinuum $Z \subset Y$ such that $z \in Z$, there is a sequence of subcontinua $Z_n \subset Y$, $z_n \in Z_n$, that converge to Z (in the sense of the Hausdorff distance).

One can easily observe that the proof of the strong $F_{\sigma\delta}$ -universality of $Col(I^n)$ presented above applies directly to family $\mathcal{K}(I^n)$, i.e., (4.7) is satisfied if $Col(I^n)$ is substituted with $\mathcal{K}(I^n)$. Moreover, if X is a locally connected continuum each of whose non-empty open subset contains a copy of I^n , $n \geq 2$, then $\mathcal{K}(X)$ is strongly $F_{\sigma\delta}$ -universal in $C(X)$ because Approach II 3.2 applies to $\mathcal{K}(X)$ with mappings φ_U being compositions of ξ (4.5) with embeddings $I^n \hookrightarrow U$. Exactly the same observation concerns families $Apo(X)$ and $LC(X)$ of locally connected subcontinua of X . We do not know if $\mathcal{K}(I^n)$ is contained in a σZ -set in $C(I^n)$ but if we restrict the family to decomposable continua, then the condition is satisfied for $n \geq 3$, since $D(I^n)$ is a σZ -set in $C(I^n)$ (Proposition 4.2). Concerning the more general case of a Peano continuum X as above, it is not known if decomposable subcontinua of X form a σZ -set in $C(X)$, so, instead, we can restrict $\mathcal{K}(X)$ (and $Apo(X)$ and $LC(X)$) to $\mathcal{D}_2(X)$ which is a σZ -set in $C(X)$ by Example 1.1(1). Summarizing, we get the following theorem.

Theorem 4.4. *Let X be a locally connected continuum each of whose non-empty open subset contains a copy of I^n , $n \geq 2$. Then $\mathcal{K}(X)$, $Apo(X)$ and $LC(X)$ are strongly $F_{\sigma\delta}$ -universal in $C(X)$. The families $\mathcal{K}(X) \cap \mathcal{D}_2(X)$, $Apo(X) \cap \mathcal{D}_2(X)$, $LC(X) \cap \mathcal{D}_2(X)$ are $F_{\sigma\delta}$ -absorbers in $C(X)$ and $\mathcal{K}(I^n) \cap D(I^n)$ is an $F_{\sigma\delta}$ -absorber in $C(I^n)$ for $n \geq 3$.*

Remark 4.5. The more general Approach II 3.2 cannot be directly applied for $Col(X)$ with mappings φ_U taken as copies of the reduction ξ (4.5), since the 1-dimensional part $A(q)$ of $g(q)$ can spoil the colocal connectedness of $g(q)$ and the remedy of surrounding it by a small closed ball (similarly as in the proof of Theorem 4.3) may kill the “one-to one” property of $g(q)$.

5. CLOSED SEPARATORS OF I^n

Proposition 5.1. *If X is a locally connected continuum, then the family $\mathcal{S}(X)$ of all compact separators of X is an F_σ -subset of 2^X .*

Proof. Since X is locally connected, a closed subset S separates X if and only if S cuts X between two points. Let \mathcal{E} be a countable dense subset of the family $\{(C, x, y) \in C(X) \times X^2 : x \neq y, x, y \in C\}$. Denote by $proj_1$ and $proj_2$ the projections of $C(X) \times X^2$ onto $C(X)$ and X^2 , respectively. We can now express the definition of a closed separator (= cut) using \mathcal{E} :

S is a closed separator of X if and only if

$$(5.1) \quad \exists(x, y) \in proj_2(\mathcal{E}) \quad (\{x, y\} \cap S = \emptyset) \quad \text{and} \\ \forall C \in proj_1(\mathcal{E} \cap (proj_2)^{-1}(x, y)) \quad (C \cap S \neq \emptyset).$$

Since the two quantifiers in (5.1) are taken over countable sets, the conclusion follows. \square

Proposition 5.2. *If a space X contains an open subset homeomorphic to the combinatorial interior $\text{int}(I^n) = (0, 1)^n$ of I^n , $2 \leq n < \infty$, then $\mathcal{S}(X) \cap C(X)$ is F_σ -universal.*

Proof. We can assume, without loss of generality, that $I^n \subset X$ and $(0, 1)^n$ is open in X . Since the pseudo-boundary $B(I^\omega)$ is an F_σ -absorber, it is enough to construct an embedding $\Psi : I^\omega \rightarrow C(X)$ such that

$$(5.2) \quad \Psi((q_i)) \in \mathcal{S}(X) \quad \text{if and only if} \quad (q_i) \in B(I^\omega).$$

Denote $J_i = [\frac{1}{2i+1}, \frac{1}{2i}]$ and let $\partial(\Delta_i)$ be the combinatorial boundary of the cube

$$\Delta_i = J_i \times \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{4i(i+1)} \right]^{n-1}, \quad i = 1, 2, \dots$$

For each i , there is a deformation $h_i(A, t)$ of 2^{Δ_i} through finite sets in

$$\left[\frac{1}{2i+1} + t, \frac{1}{2i} - t \right] \times \left[\frac{1}{2} + t, \frac{1}{2} + \frac{1}{4i(i+1)} - t \right]^{n-1}, \quad 0 < t < \frac{1}{8i(i+1)}$$

(look at h_i as H_0 considered in Section 3 with I^n replaced by Δ_i). Given $t \in I$, choose points

$$x_i(t) = \left(\frac{1}{2i+1} + t \frac{1}{4i(i+1)}, \frac{1}{2}, \dots, \frac{1}{2} \right)$$

and

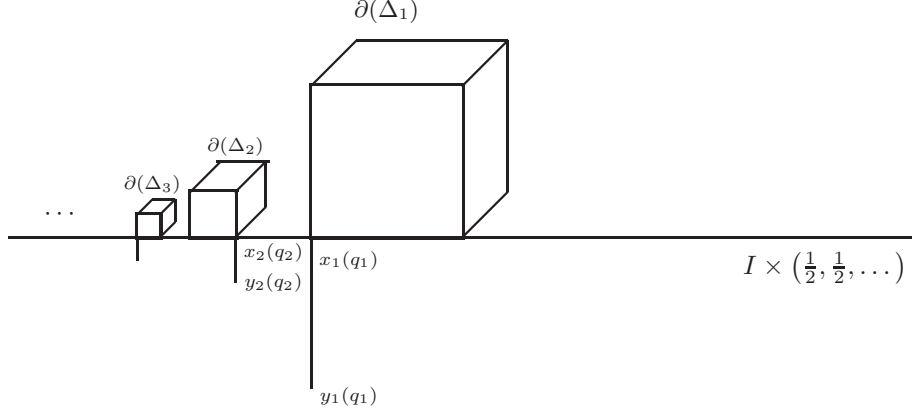
$$y_i(t) = \left(\frac{1}{2i+1} + t \frac{1}{4i(i+1)}, \frac{1}{2} - \frac{1}{4i(i+1)}, \frac{1}{2}, \dots, \frac{1}{2} \right).$$

For $0 < t < \frac{1}{8i(i+1)}$ and each i , connect points of the set $h_i(A, t) \cup \{x_i\}$ following formula (3.2):

$$(5.3) \quad H_i(A, t) = \bigcup_{a, b \in h_i(A, t) \cup \{x_i\}} (\overline{ab} \cap (\overline{B(a; 2t)} \cup \overline{B(b; 2t)}))$$

and put $H_i(A, 0) = A$. Thus H_i is a deformation 2^{Δ_i} through finite graphs in Δ_i that meet the edge $J_i \times \{(\frac{1}{2}, \dots, \frac{1}{2})\}$ at the single point x_i . The function $\Psi : I^\omega \rightarrow C(I^n)$ defined by

$$(5.4) \quad \Psi((q_i)) = \\ I \times \left\{ \left(\frac{1}{2}, \dots, \frac{1}{2} \right) \right\} \cup \bigcup_i H_i \left(\partial(\Delta_i), \frac{q_i(1 - q_i)}{8i(i+1)} \right) \cup \bigcup_i \overline{x_i(q_i) y_i(q_i)}$$

FIGURE 3. $\Psi((0, 1, 0, \dots))$

(see Figure 3) is continuous and it is 1-1 since the correspondence $(q_i) \mapsto \overline{(x_i(q_i)y_i(q_i))}$ is 1-1. It satisfies (5.2) for $n \geq 3$ because if $(q_i) \in B(I^\omega)$, then $\Psi((q_i))$ contains $\partial(\Delta_i)$ for some i which separates $\text{int}(I^n)$ — consequently, by construction, $\Psi((q_i))$ separates X ; otherwise, $\Psi((q_i))$ is one-dimensional, so it does not separate I^n , $n \geq 3$.

If $n = 2$, the above construction does not work, since $\Psi((q_i))$ might separate I^2 for (q_i) in the pseudo-interior s . In this case we can, however, define an appropriate embedding Ψ differently much easier. Given $(q_i) \in I^\omega$, denote $p_{2i} = x_i(q_i)$, $p_{2i-1} = x_i(1 - q_i)$, $d_i = (\frac{1}{2i+1}, \frac{1}{2})$, $A_i = \overline{d_i p_i} \setminus \{d_i, p_i\}$. Put

$$(5.5) \quad \Psi((q_i)) = I \times \left\{ \frac{1}{2} \right\} \cup \bigcup_i (\partial(\Delta_i) \setminus A_i).$$

□

Theorem 5.3. *Let X be a locally connected continuum such that each open non-empty subset of X contains a copy of $(0, 1)^n$, $3 \leq n < \infty$, as an open subset and no subset of dimension ≤ 1 separates X . Then the families $\mathcal{S}(X)$ and $\mathcal{S}(X) \cap C(X)$ are F_σ -absorbers in 2^X and $C(X)$, respectively.*

Proof. Since being an F_σ -absorber in a Hilbert cube is equivalent, for an F_σ -set, to being strongly F_σ -universal [2, Theorem 5.3], it remains to prove the strong F_σ -universality. To this end, we are going to use Approach II 3.2. The pseudo-boundary $B(I^\omega)$ being strongly F_σ -universal, there exists for each $M \subset I^\omega$, $M \in F_\sigma$, a mapping $\chi : I^\omega \rightarrow I^\omega$ such that $\chi^{-1}(B(I^\omega)) = M$. Hence, the composition $\varphi = \Psi\chi : I^\omega \rightarrow C(I^n)$ (Ψ as in (5.4)) satisfies $\varphi^{-1}(\mathcal{S}(I^n)) = M$ for every $q \in I^\omega$. For each open non-empty subset U of X and an open copy of $(0, 1)^n$ in U , let $\varphi_U : I^\omega \rightarrow C(U)$ be a composition of φ with an embedding of $C(I^n)$ into the hyperspace of that copy. Then $\varphi_U(q)$ separates X iff $\varphi(q)$ separates I^n iff $q \in B(I^\omega)$.

The properties of g in Approach II 3.2 yield that g satisfies (3.1) for $\mathcal{A} = \mathcal{S}(X)$ as well as for $\mathcal{A} = \mathcal{S}(X) \cap C(X)$. In fact, for $q \notin K$, $g(q)$ separates X if and only if $\varphi_U(q)$ does, because neither the zero- nor one-dimensional part $A(q)$ of $g(q)$ do affect separation of X by copies $\varphi_U(q)$ for $n \geq 3$. This completes the proof. □

Corollary 5.4. *The families $\mathcal{S}(X)$ and $\mathcal{S}(X) \cap C(X)$ are F_σ -absorbers in 2^X and $C(X)$, respectively, if X is a continuum which is an n -manifold (with or without boundary), $3 \leq n < \infty$.*

Denote by $\mathcal{N}(X)$ the family of all nowhere dense closed subsets of X . The following proposition is well known and has a straightforward proof.

Proposition 5.5. *For any compact space X , the subspace of 2^X consisting of all closed subsets of X with non-empty interiors is an F_σ -set.*

The next fact follows from Propositions 5.1 and 5.5.

Proposition 5.6. *If X is a locally connected continuum, then*

$$\mathcal{S}(X) \cap \mathcal{N}(X) \in D_2(F_\sigma) \quad \text{and} \quad \mathcal{S}(X) \cap \mathcal{N}(X) \cap C(X) \in D_2(F_\sigma).$$

Proposition 5.7. *If a space X contains an open subset homeomorphic to the combinatorial interior $\text{int}(I^n) = (0, 1)^n$ of I^n , $2 \leq n < \infty$, then $\mathcal{S}(X) \cap \mathcal{N}(X) \cap C(X)$ is $D_2(F_\sigma)$ -universal.*

Proof. Assume again that $I^n \subset X$ and $(0, 1)^n$ is an open subset of X . First, replace in the definition (5.4) of embedding Ψ the boundaries $\partial(\Delta_i)$ with the cubes Δ_i themselves for all i 's and denote thus obtained embedding by Ψ_0 . We have

$$(5.6) \quad \Psi_0 : I^\omega \rightarrow C(I^n) \quad \text{and} \quad \Psi_0^{-1}(\mathcal{N}(X)) = s,$$

where s is the pseudo-interior of I^ω .

Let $\alpha : I^n \rightarrow [-1, 0] \times I^{n-1}$ be the reflection

$$(5.7) \quad \alpha(x_1, x_2, \dots, x_n) = (-x_1, x_2, \dots, x_n).$$

Fix an embedding $f : [-1, 1] \times I^{n-1} \rightarrow \text{int}(I^n) \subset X$. Define a continuous embedding

$$(5.8) \quad \Phi : I^\omega \times I^\omega \rightarrow C(X), \quad \Phi((q_i), (t_i)) = f(\Psi((q_i)) \cup \alpha(\Psi_0((t_i)))).$$

For $n \geq 3$, it follows from (5.2) and (5.6) that

$$(5.9) \quad \Phi((q_i), (t_i)) \in \mathcal{S}(X) \cap \mathcal{N}(X) \cap C(X) \quad \text{if and only if} \\ ((q_i), (t_i)) \in B(I^\omega) \times s.$$

In case $n = 2$ we define map Φ by formula (5.8) in which Ψ is the map from (5.5) and ϕ is defined in the following way.

Let $f_i : 2^{J_i} \times I \rightarrow 2^{J_i}$ be a deformation through finite sets. Denote

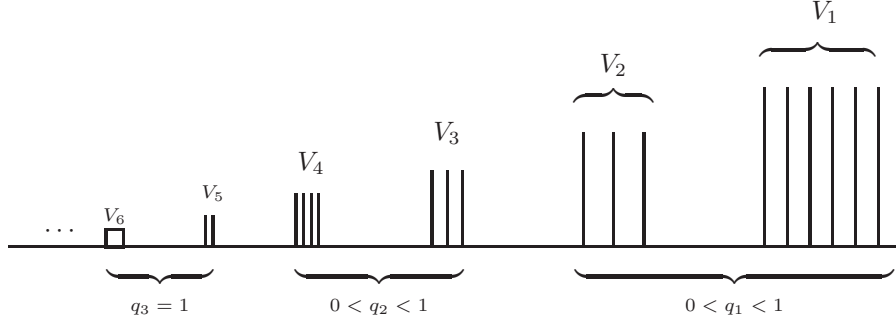
$$W_i(t) = f_i(J_i, t) \times \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{4i(i+1)} \right], \\ V_{2i-1} = W_i((q_i)), \quad V_{2i} = W_i((1 - q_i))$$

for $(q_i) \in I^\omega$. Observe that $W_i(0) = \Delta_i$. Now, the map

$$\phi((q_i)) = I \times \left\{ \frac{1}{2} \right\} \cup \bigcup_i V_i$$

is an embedding which satisfies (5.9) (Figure 4).

Since $B(I^\omega) \times s$ is strongly $D_2(F_\sigma)$ -universal (see Example 1.3), the proof is complete. \square

FIGURE 4. $\phi((q_1, q_2, 1, \dots))$ in I^2

Theorem 5.8. *Assume X satisfies hypotheses of Theorem 5.3. Then $\mathcal{S}(X) \cap \mathcal{N}(X)$ is an $D_2(F_\sigma)$ -absorber in 2^X and $\mathcal{S}(X) \cap \mathcal{N}(X) \cap C(X)$ is an $D_2(F_\sigma)$ -absorber in $C(X)$ for $n \geq 3$.*

Proof. Since

$$\mathcal{S}(X) \cap \mathcal{N}(X) \subset \mathcal{S}(X), \quad \mathcal{S}(X) \cap \mathcal{N}(X) \cap C(X) \subset \mathcal{S}(X) \cap C(X)$$

and $\mathcal{S}(X)$ and $\mathcal{S}(X) \cap C(X)$ are σZ -sets in 2^X and $C(X)$, respectively (by Theorem 5.3), it remains to show the strong $D_2(F_\sigma)$ -universality of both families. There exists, for each subset $M \subset I^\omega$ from the class $D_2(F_\sigma)$, a mapping

$$\zeta : I^\omega \rightarrow I^\omega \times I^\omega \quad \text{such that} \quad \zeta^{-1}(B(I^\omega) \times s) = M.$$

The composition $\varphi = \Phi\zeta$, where Φ is the mapping (5.8), provides a map $I^\omega \rightarrow C(I^n) \subset C(X)$ satisfying

$$\varphi^{-1}((\mathcal{S}(X) \cap \mathcal{N}(X) \cap C(X))) = M$$

and mappings $\varphi_U : I^\omega \rightarrow C(U)$, for every open non-empty subset U of X and an open copy of $(0, 1)^n$ in U , which are compositions of φ with an embedding of $C(I^n)$ into the hyperspace of that copy. Now, the construction of the embedding g in Approach 3.2 works for both families, i.e., if $q \notin K$, then

$$g(q) \in \mathcal{S}(X) \cap \mathcal{N}(X) \cap C(X) \quad \text{if and only if}$$

$$\varphi(q) \in \mathcal{S}(I^n) \cap \mathcal{N}(I^n) \cap C(I^n)$$

and similarly for $\mathcal{S}(X) \cap \mathcal{N}(X)$, because neither the zero- nor one-dimensional part $A(q)$ of $g(q)$ destroy the separation properties of copies $\varphi_U(q)$ in $g(q)$ or change their status of being nowhere dense in X for $n \geq 3$. □

Corollary 5.9. *If X is a continuum which is an n -manifold (with or without boundary), $3 \leq n < \infty$, then the family $\mathcal{S}(X)_{n-1}$ of all $(n-1)$ -dimensional closed separators of X is a $D_2(F_\sigma)$ -absorber in 2^X and $\mathcal{S}(X)_{n-1} \cap C(X)$ is a $D_2(F_\sigma)$ -absorber in $C(X)$.*

Proof. It is well known that each such X is a Cantor n -manifold (\equiv no subset of dimension $\leq n-2$ separates X) and $Y \in \mathcal{N}(X)$ iff $\dim Y \leq n-1$. This means that

$$\mathcal{S}(X) \cap \mathcal{N}(X) = \mathcal{S}(X)_{n-1}.$$

□

6. INFINITE-DIMENSIONAL COMPACTA

Recall that a space X is *strongly infinite-dimensional* if there exists a sequence $(A_n, B_n)_n$ of closed disjoint subsets of X such that for each sequence $(C_n)_n$ of closed separators of X between A_n and B_n we have $\bigcap_n C_n \neq \emptyset$.

A space is *weakly infinite-dimensional* if it is not strongly infinite-dimensional. The collection of all weakly infinite-dimensional compacta in a space Y will be denoted by $\mathcal{W}(Y)$.

It was proved in [26, Section 4, p. 173] that strongly infinite-dimensional compacta form an analytic subset of 2^{I^ω} . Below, we provide a different and elementary proof of this fact for such compacta in an arbitrary locally connected compact space.

Proposition 6.1. *The family of strongly infinite-dimensional compacta in a compact locally connected space Y is an analytic subset of 2^Y .*

Proof. In view of Observation 2.1, we have the following claim.

Claim 6.1.1. A compact space $X \subset Y$ is strongly infinite-dimensional if and only if

$$(6.1) \quad \exists (A_n, B_n)_n \in (2^Y \times 2^Y)^\omega \\ \forall n (A_n \subset X, B_n \subset X, A_n \cap B_n = \emptyset) \quad \text{and} \quad \forall (C_n) \in (2^Y)^\omega \\ (\text{if } \forall n (C_n \text{ cuts } Y \text{ between } A_n \text{ and } B_n), \text{ then } X \cap \bigcap_n C_n \neq \emptyset).$$

A rough evaluation of the projective complexity of formula (6.1) gives merely class Π_2^1 . Therefore we need to refine the cutting condition in (6.1).

Let \mathcal{E} be a countable dense subset of the family

$$\{(C, A, B) \in (2^Y)^3 : A \cap B = \emptyset, \quad C \text{ cuts } Y \text{ between } A \text{ and } B\}$$

and let $\mathcal{E}_1 = \text{proj}_1(\mathcal{E})$, where proj_1 is the projection of $(2^Y)^3$ onto the first factor-space.

Claim 6.1.2. A compact space $X \subset Y$ is strongly infinite-dimensional if and only if

$$(6.2) \quad \exists (A_n, B_n)_n \in (2^Y \times 2^Y)^\omega \\ \forall n (A_n \subset X, B_n \subset X, A_n \cap B_n = \emptyset) \quad \text{and} \quad \forall k \forall (C_1, \dots, C_k) \in (\mathcal{E}_1)^k \\ (\text{if } \forall i \leq k (C_i \text{ cuts } Y \text{ between } A_i \text{ and } B_i), \text{ then } X \cap \bigcap_{i \leq k} C_i \neq \emptyset).$$

In order to show the less obvious implication \Leftarrow , assume that C_n cuts Y between A_n and B_n for each $n \in \mathbb{N}$. For each k , approximate (C_i, A_i, B_i) by $(C'_i, A'_i, B'_i) \in \mathcal{E}$, $i \leq k$. Then, by the local connectedness of Y , C'_i cuts Y between A_i and B_i if the approximation is sufficiently close. Then $X \cap \bigcap_{i \leq k} C'_i \neq \emptyset$. So, there is a point $x_k \in X \cap \bigcap_{i \leq k} C'_i$ and the distances $d(x_k, C_1), \dots, d(x_k, C_k)$ can be made arbitrarily small. Let x be an accumulation point of sequence (x_k) . Then $x \in X \cap \bigcap_{i \leq k} C_i$ and, by the compactness of Y , we get $X \cap \bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$ which completes the proof of Claim 6.1.2.

We are now ready to evaluate the complexity of formula (6.2). First, write down the sentence

C_i cuts Y between A_i and B_i

as

$$(6.3) \quad C_i \cap A_i = \emptyset, \quad C_i \cap B_i = \emptyset \quad \text{and} \quad \forall D \in C(Y) \\ (D \cap A_i \neq \emptyset, \quad D \cap B_i \neq \emptyset) \Rightarrow D \cap C_i \neq \emptyset$$

and observe that its Borel complexity is G_δ . Next, notice that all quantifiers in formula (6.2), preceding the sentence, except for the first existential one, are taken over at most countable sets of variables, hence the formula following the first existential quantifier describes a Borel set. Now, the whole formula (6.2) gives an analytic set, as a continuous projection of a Borel set. \square

Proposition 6.2. *Let Y be a compact, locally connected space containing a Hilbert cube. Then, for each integer $n \geq 0$, the subsets $\mathcal{W}_n(Y)$ of 2^Y and $\mathcal{W}_n(Y) \cap C(Y)$ of $C(Y)$ consisting of weakly infinite-dimensional compacta and continua, respectively, of dimensions $\geq n$ are Π_1^1 -complete.*

Proof. It is well known that the family $\mathcal{D}_n(Y)$ of all closed subsets of Y that have dimension at least n is F_σ in 2^Y (for any compact Y). Since $\mathcal{W}_n(Y) = \mathcal{W}(Y) \cap \mathcal{D}_n(Y)$, the set $\mathcal{W}_n(Y)$ is coanalytic by Proposition 6.1. Hence $\mathcal{W}_n(Y) \cap C(Y)$ is also coanalytic. The Π_1^1 -hardness of $\mathcal{W}(Y) \cap C(Y)$ for $Y = I^\omega$ was established in [18, Corollary 3.3] and an analogous argument gives the hardness for each set $\mathcal{W}_n(Y) \cap C(Y)$ and arbitrary Y as in the hypothesis. It follows immediately that each $\mathcal{W}_n(Y)$ is Π_1^1 -hard as well. \square

Recall that a space X is a C -space if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of open covers of X there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \dots$ of families of pairwise disjoint open subsets of X such that each \mathcal{V}_i refines \mathcal{U}_i and $\bigcup_{i=1}^\infty \mathcal{V}_i$ is a cover of X .

If (X, d) is compact, then covers $\mathcal{U}_1, \mathcal{U}_2, \dots$ can be replaced by a sequence of positive reals $\epsilon_i \rightarrow 0$, if $i \rightarrow \infty$, and the cover $\bigcup_{i=1}^\infty \mathcal{V}_i$ can be replaced by a finite subcover. Each family \mathcal{V}_i is then finite and its elements have diameters $< \epsilon_i$. The definition can be rewritten as follows:

$$(6.4) \quad \forall (n_1, n_2, \dots) \in \mathbb{N}^\omega \quad \exists k \in \mathbb{N} \quad \exists (\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_k) \quad \forall i \leq k \\ [\mathcal{V}_i \text{ is finite, consists of open subsets of } X \text{ and} \\ [V, V' \in \mathcal{V}_i, V \neq V' \Rightarrow V \cap V' = \emptyset] \\ \text{and} \quad [V \in \mathcal{V}_i, x, y \in V \Rightarrow d(x, y) < \frac{1}{n_i}] \quad \text{and} \quad \bigcup_{i \leq k} \mathcal{V}_i = X].$$

Proposition 6.3. *For each $n \geq 0$, the families $\mathcal{C}_n(Y)$ of C -compacta of dimensions $\geq n$ in a compact space Y and $\mathcal{C}_n(Y) \cap C(Y)$ are coanalytic subsets of 2^Y .*

Proof. In view of (6.4), the definition of a C -compactum (X, d) in terms of closed subsets of Y runs as follows:

$$(6.5) \quad \begin{aligned} & \forall (n_1, n_2, \dots) \in \mathbb{N}^\omega \exists k \in \mathbb{N} \exists (\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k) \in (2^{(2^Y)})^k \\ & \quad \forall i \leq k \left[\mathcal{F}_i \text{ is finite and } \forall F \in \mathcal{F}_i (F \subset X) \right. \\ & \quad \text{and } \forall F, F' \in \mathcal{F}_i (F \neq F' \Rightarrow F \cup F' = X) \\ & \quad \left. \text{and } \forall F \in \mathcal{F}_i (x, y \in X \setminus F \Rightarrow d(x, y) < \frac{1}{n_i}) \text{ and } \bigcap_{i \leq k} \mathcal{F}_i = \emptyset \right]. \end{aligned}$$

Formula (6.5) easily yields that the set $\mathcal{C}(Y)$ of all C -compacta in Y is coanalytic. Hence, each of $\mathcal{C}_n(Y) = \mathcal{C}(Y) \cap \mathcal{D}_n(Y)$ and $\mathcal{C}_n(Y) \cap C(Y)$ is also coanalytic. \square

Proposition 6.4. *Let Y be a compact space containing a Hilbert cube. Then, for each integer $n \geq 0$, the families $\mathcal{C}_n(Y)$ and $\mathcal{C}_n(Y) \cap C(Y)$ are Π_1^1 -complete.*

Proof. The argument for the Π_1^1 -hardness of each of the above families is the same as in the proof of Proposition 6.2. So, the conclusion follows from Proposition 6.3. \square

Proposition 6.5. *Let Y be a locally connected continuum.*

- (1) *The families $\mathcal{W}_n(Y)$, $\mathcal{W}_n(Y) \cap C(Y)$, $\mathcal{C}_n(Y)$ and $\mathcal{C}_n(Y) \cap C(Y)$, $n \geq 1$, are contained in $\mathcal{D}_1(Y)$, a σZ -set in 2^Y .*
- (2) *If each non-empty open subset of Y contains an n -cell, $n \geq 2$, then the families $\mathcal{W}_n(Y) \cap C(Y)$ and $\mathcal{C}_n(Y) \cap C(Y)$ are contained in $\mathcal{D}_2(Y) \cap C(Y)$, a σZ -set in $C(Y)$.*

Proof. The set $\mathcal{D}_1(Y)$ is a σZ -set in 2^Y since there is a deformation $2^Y \times I \rightarrow 2^Y$ through finite sets [8]. The set $\mathcal{D}_2(Y) \cap C(Y)$ is a σZ -set in $C(Y)$ as it is an F_σ -absorber in $C(Y)$ (see Examples 1.1 (1)). \square

Denote by $\mathcal{SCD}_n(X)$ the family of all strongly countable-dimensional compacta of dimension $\geq n$ in a space X .

Theorem 6.6. *Let X be a locally connected continuum such that each non-empty open subset of X contains a copy of the Hilbert cube (in particular, X can be a Hilbert cube manifold).*

Families $\mathcal{SCD}_n(X)$, $\mathcal{W}_n(X)$, and $\mathcal{C}_n(X)$ are coanalytic absorbers in 2^X for $n \geq 1$.

Families $\mathcal{SCD}_n(X) \cap C(X)$, $\mathcal{W}_n(X) \cap C(X)$ and $\mathcal{C}_n(X) \cap C(X)$ are coanalytic absorbers in $C(X)$ for $n \geq 2$.

Proof. We have $\mathcal{SCD}_n(X) \subset \mathcal{C}_n(X) \subset \mathcal{W}_n(X)$ (see [13]). In view of Propositions 6.2, 6.3 and 6.5, it suffices to check the strong Π_1^1 -universality of the families. In the case when $X = I^\omega$ an Approach I-construction used in the proof of [20, Theorem 3.1] for the strong Π_1^1 -universality of $\mathcal{SCD}_n(I^\omega)$ applies to other families without any change. Its main ingredient is a continuous mapping $\xi : I^\omega \rightarrow C(I^\omega)$ such that

- $\xi(q)$ is a strongly countable-dimensional continuum of dimension $\geq n$ for $q \in M$ (actually, it is a countable union of Euclidean cubes of dimensions $\geq n$) and

- $\xi(q)$ contains a Hilbert cube for $q \notin M$,

where M is a coanalytic subset of I^ω . In other words, ξ is a continuous reduction of M to $SCD_n \cap C(I^\omega)$.

Observe that ξ can also be viewed as a reduction of M to any of the families in our theorem, since their members do not contain Hilbert cubes.

In a general case, we use Approach II 3.2. So, for each nonempty open subset U of X , choose an embedding $\eta_U : I^\omega \rightarrow U$ and let $\varphi_U(q) = \eta_U(\xi(q))$. We have mappings $\varphi_U : I^\omega \rightarrow C(U)$ satisfying $\varphi_U^{-1}(\mathcal{A}) = M$, where \mathcal{A} is any of the considered families. Since taking finite unions of copies $\varphi_U(q)$ and adding zero- or one-dimensional part $A(q)$ of $g(q)$ do not change dimension properties of members of each family, we have that $g(q) \in \mathcal{A}$ iff $q \in M$ for $q \in I^\omega \setminus K$. \square

Other intriguing spaces studied in dimension theory are *hereditarily infinite-dimensional compacta* in the sense of Henderson, i.e., infinite-dimensional compact spaces whose nonempty closed subspaces are either infinite-dimensional or zero-dimensional.

Proposition 6.7. *The collection $\mathcal{HID}(Y)$ of all hereditarily infinite-dimensional compacta in a compact space Y is a coanalytic subset of 2^Y .*

Proof. We have $X \in \mathcal{HID}(Y)$ if and only if

$$(6.6) \quad \forall F \in 2^Y \text{ (if } F \subset X, \text{ then } \dim F = 0 \text{ or } \forall n \dim F \geq n).$$

A direct evaluation of the projective complexity of (6.6) shows that X is coanalytic. \square

Theorem 6.8. *The families $\mathcal{HID}(I^\omega)$ and $\mathcal{HID}(I^\omega) \cap C(I^\omega)$ are Π_1^1 -complete subsets of 2^{I^ω} and of $C(I^\omega)$, respectively.*

Proof. The proof of Π_1^1 -hardness will use an idea of R. Pol presented in [25, Lemma 6.3].

Take a continuum $L \in \mathcal{HID}(I^\omega)$ and the set \mathcal{L} of all topological copies of L in I^ω . Let \mathcal{P} be the collection of all pseudoarcs in I^ω . Families \mathcal{P} and \mathcal{L} are dense in $C(I^\omega)$ and \mathcal{P} is a G_δ -subset of $C(I^\omega)$. Let Q be a countable dense subset of a Cantor set $C \subset I$.

Let $\varphi : C \rightarrow \{\mathcal{P}, \mathcal{L}\}$ be a function defined by

$$\varphi(x) = \begin{cases} \mathcal{P}, & \text{for } x \in C \setminus Q; \\ \mathcal{L}, & \text{for } x \in Q. \end{cases}$$

By [23, Corollary 1.61], $\varphi(x)$ has a continuous selection

$$\sigma : C \rightarrow C(I^\omega), \quad \sigma(x) \in \varphi(x).$$

Define the map $\pi : 2^C \rightarrow 2^{I \times I^\omega} \stackrel{\text{top}}{\cong} 2^{I^\omega}$ by

$$\pi(A) = \bigcup_{x \in A} \{x\} \times \sigma(x).$$

Observe that $\pi^{-1}(\mathcal{HID}(I^\omega)) = 2^Q$ is the Hurewicz set of all compacta in Q , a standard coanalytic complete set. In other words, π is a continuous reduction of 2^Q to $\mathcal{HID}(I^\omega)$.

If we identify to a point the Cantor set level of each $\pi(A)$ and respectively modify π , then we get a continuous reduction of 2^Q to $\mathcal{HID}(I^\omega) \cap C(I^\omega)$. \square

Families $\mathcal{HID}(I^\omega)$ and $\mathcal{HID}(I^\omega) \cap C(I^\omega)$ are contained, respectively, in $\mathcal{D}_2(I^\omega)$ and $\mathcal{D}_2(I^\omega) \cap C(I^\omega)$ which are σZ -sets in 2^{I^ω} and in $C(I^\omega)$.

We do not know if $\mathcal{HID}(I^\omega)$ is strongly Π_1^1 -universal in 2^{I^ω} .

REFERENCES

1. T. Banach, T. Radul and M. Zarichnyi, Absorbing sets in Infinite-Dimensional Manifolds, VNTL Publishers, Lviv, 1996.
2. J. Baars, H. Gladdines and J. van Mill, Absorbing systems in infinite-dimensional manifolds, *Topology Appl.* 50 (1993), 147-182.
3. R. Cauty, Caractérisation topologique de l'espace des fonctions dérivables, *Fund. Math.* 138 (1991), 35-58.
4. R. Cauty, L'espace des arcs d'une surface, *Trans. Amer. Math. Soc.* 332 (1992), 193-209.
5. R. Cauty, Suites F_σ -absorbantes en théorie de la dimension, *Fund. Math.* 159 (1999), 115-126.
6. R. Cauty, T. Dobrowolski, H. Gladdines and J. van Mill, Les hyperespaces des rétractes absolus et des rétractes absolus de voisinage du plan, *Fund. Math.* 148 (1995), 257-282.
7. T. A. Chapman, Lectures on Hilbert cube manifolds, AMS, Providence, Rhode Island, 1976.
8. D. W. Curtis, Hyperspaces of finite subsets as boundary sets, *Topology Appl.* 22 (1986), 97-107.
9. D. Curtis and M. Michael, Boundary sets for growth hyperspaces, *Topology Appl.* 25 (1987), 269-283.
10. D. Curtis and M. Schori, Hyperspaces of Peano continuaare Hilbert cubes, *Fund. Math.* 101 (1978), 19-38.
11. J. J. Dijkstra, J. van Mill and J. Mogilski, The space of infinite-dimensional compacta and other topological copies of $(l_f^2)^\omega$, *Pacific J. Math.* 152 (1992), 255-273.
12. T. Dobrowolski and L. R. Rubin, The space of ANRs in \mathbf{R}^n , *Fund. Math.* 146 (1994), 31-58.
13. R. Engelking, Theory of dimensions: Finite and Infinite, Heldermann Verlag, 1995.
14. H. Gladdines and J. van Mill, Hyperspaces of Peano continua of Euclidean spaces, *Fund. Math.* 142 (1993), 173-188.
15. A. Illanes and P. Krupski, Blockers in hyperspaces, *Topology Appl.* 158 (2011), 653-659.
16. A. Kechris, Classical descriptive set theory, Springer, 1995.
17. K. Króllicki and P. Krupski, Wilder continua and their subfamilies as coanalytic absorbers, *Topology Appl.* (to appear), arXiv:1512.05802
18. P. Krupski, More non-analytic classes of continua, *Topology Appl.* 127 (2003), 299-312.
19. P. Krupski, Families of continua with the property of Kelley, arc continua and curves of pseudo-arcs, *Houston J. Math.* 30 (2004), 459-482.
20. P. Krupski and A. Samulewicz, Strongly countable dimensional compacta form the Hurewicz set, *Topology Appl.* 154 (2007), 996-1001.
21. K. Kuratowski, *Topology*. Vol. I, Academic Press-PWN, 1966.
22. K. Kuratowski, *Topology*. Vol. II, Academic Press-PWN, 1968.
23. E. Michael, Some refinements of a selection theorem with 0-dimensional domain, *Fund. Math.* 140 (1992), 279-287.
24. J. van Mill, *The Infinite-Dimensional Topology of Function Spaces*, North-Holland, 2002.
25. E. Pol, On infinite-dimensional Cantor manifolds, *Topology Appl.* 71 (1996), 265-276.
26. R. Pol, On classification of weakly infinite-dimensional compacta, *Fund. Math.* 116 (1983), 169-188.
27. A. Samulewicz, The hyperspace of hereditarily decomposable subcontinua of a cube is the Hurewicz set, *Topology Appl.* 154 (2007), 985-995.
28. A. Samulewicz, The hyperspace of indecomposable subcontinua of a cube, *Bol. Soc. Mat. Mexicana* (3) 17 (2011), no. 1, 89-91.
29. B. E. Wilder, Between aposyndetic and indecomposable continua, *Topology Proc.* 17 (1992), 325-331.

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